New Types of Solitons with Fusion and Fission Properties in the (2+1)-Dimensional Generalized Broer-Kaup System

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In this paper, by virtue of a special Painlevé-Bäcklund transformation and the linear superposition theorem, the general variable separation solution with an arbitrary number of variable separated functions of the generalized Broer-Kaup (GBK) system is obtained. Based on the general variable separation solution with some suitable variable separated functions, new types of the V-shaped soliton fusion and Y-shaped soliton fission are firstly investigated. – PACS numbers: 05.45.Yv, 02.30.Jr, 02.03Ik

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1. Introduction

In linear physics, it is generally recognized that the variable separation approach (VSA) is one of the most universal and powerful means for the study of linear partial differential equations (PDEs). The extension of the to nonlinear field has been a highlight. Recently, the so-called multilinear variable separation approach (MLVSA) has been established [1]. More recently, the extended tanh-function method (ETM) based on mapping method [2] and the projective Riccati equation method (PREM) [3] are generalized to derive the variable separation solutions of a large type of nonlinear evolutional equations (NEEs).

In the linear case, any number of variable separation solutions can be linearly superposed to yield new solutions of the same system. Thus, naturally, there comes an important question, whether this idea can be extended in a way that more variable separated functions are included in the variable separation solutions of NEEs. In this paper, by means of a special Painlevé-Bäcklund transformation and the linear superposition theorem, a general variable separation solution with an arbitrary number of variable separated functions is obtained to the following generalized Broer-Kaup (GBK) system:

$$
\begin{aligned}
&h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0,\\
&g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy})\\
&\quad + 4B(g_y - h_{yy}) + C(g - 2h_y) = 0,\\
&u_y - g_x = 0,
\end{aligned}
\qquad (1)
$$

where A, B, C are arbitrary constants. The GBK system is recently derived from a typical (1+1)-dimensional Broer-Kaup (BK) system [4], by means of the Painlevé analysis [5]. When $A = B = C = 0$, the GBK system will be degenerated to the celebrated (2+1)-dimensional BK system [6], which can be derived from the inner parameter-dependent symmetry constraint of the Kadomtsev-Petviashvili model [7]. Actually, the (2+1)-dimensional GBK system has been widely investigated by many researchers. Zheng et al. [8] discussed some semifolded localized coherent structures of this system. Ma et al. [9] derived some coherent exications of (1) with the help of the projective Riccati equation. Moreover, Huang and Zhang [10] and Wu et al. [11] obtained some special soliton-like solutions and abundant localized coherent excitations for this system with $B = 0$.

Rich localized coherent structures, such as lumps, dromions, peakons, compactons, foldons, and ring solitons, were discussed in [1 – 3, 8, 9, 11]. The interactions were also investigated either between single-valued solitons [1, 9] or between multi-valued solitons (foldons) [2, 8, 11, 12]. Moreover, soliton fusion and fission phenomena among peakons, compactons, dromions and semifoldons were also investigated in [3]. However, these localized coherent structures were discussed based on two variable separated functions. To the best of our knowledge, the studies on the general solution with more variable separated functions, and especially on solitons with fusion and fission properties for the new (2+1)-dimensional GBK system were not reported in the preceding literature though Tang and Lou [13] discussed the Y-shaped soliton fusion phenomenon in the (2+1)-dimensional Burgers equation. In fact, the soliton fusion and fission phenomena have been observed in many physical systems, such as in organic membrane and macromolecular material [14], in Sr-Ba-Ni oxidation crystals and waveguides [15], in even-clump DNA [16] and in many physical fields like plasma physics, nuclear physics and hydrodynamics [17].

The paper is arranged as follows. In Section 2, we obtain the general variable separation solution with an arbitrary number of variable separated functions of the GBK system via a special Painlevé-Bäcklund transformation and the linear superposition theorem. Based on the general variable separation solution with some suitable variable separated functions, new types of the V-shaped soliton fusion and Y-shaped soliton fission are firstly discussed in Section 3. Finally, a short summary is presented.

2. Variable Separation Solution for the (2+1)-Dimensional GBK System

Via the standard truncated Painlevé expansion [1], we have a special Painlevé-Bäcklund transformation for differentiable functions h, g, u in (1):

$$h = (\ln f)_x + h_0, \quad g = 2(\ln f)_{xy} + g_0, \quad u = 2(\ln f)_{xx} + u_0, \tag{2}$$

where $f = f(x,y,t)$ is an arbitrary differentiable function of variables $\{x,y,t\}$ to be determined, and h_0, g_0, u_0 are arbitrary seed solutions satisfying the GBK system. In usual cases, by choosing some special trivial solutions, we can directly obtain the seed solutions. In the present case, via some simple calculations, it is evident that (1) possesses the trivial seed solutions

$$h_0 = h_0(x,t), \quad g_0 = 0, \quad u_0 = \exp(-Ax)\left[F_1(t) + \int^x \exp(Ax') \cdot (h_{0x'x'} - 2h_0 h_{0x'} - h_{0t})\mathrm{d}x'\right], \tag{3}$$

where $h_0(x,t)$ and $F_1(t)$ are arbitrary functions of $\{x,t\}$ and t, respectively.

Substituting (2) with the seed solutions (3) into (1) yields

$$f_t + 2(A+h_0) f_x + 2B f_y + 2 f_{xx} = 0. \tag{4}$$

Since (4) is only a linear equation, one can certainly utilize the linear superposition theorem. For instance

$$f = Q_0(y-2Bt) + \sum_{i=1}^{N} P_i(x,t) Q_i(y-2Bt,t), \tag{5}$$

where the variable separated functions $P_i(x,t) \equiv P_i$ and $Q_i(y-2Bt,t) \equiv Q_i$ $(i = 1,2,\cdots,N)$ are only the functions of $\{x,t\}$ and $\{y,t\}$, respectively, and $Q_0(y-2Bt) \equiv Q_0$. Inserting the ansatz (5) into (4) yields

$$Q_i[P_{it} + 2P_{ixx} + 2(A+h_0)P_{ix}] + P_i Q_{it} = 0. \tag{6}$$

Similarly to the linear cases, we can obtain the following simple variable separated equations:

$$P_{it} + 2P_{ixx} + 2(A+h_0)P_{ix} + b_i(t)P_i = 0, \tag{7}$$

$$Q_{it} - b_i(t)Q_i = 0, \tag{8}$$

where $b_i(t)$ $(i = 1,2,\cdots,N)$ are arbitrary functions of the indicated variable. Solving (7) is very difficult. However, because of the arbitrariness of h_0, we can treat the problem alternatively. Actually, we can consider the functions P_i as arbitrary functions while h_0 is fixed by (7). Therefore, h_0 and Q_i can be derived by solving (7) and (8), respectively, while the functions P_i are considered as arbitrary functions.

Substituting all the results into (2), we obtain the corresponding exact solution of (1). Especially we are interested in the structure of the solution for the field g which has the final form

$$g = \frac{2\sum_{i=1}^{N} P_{ix} Q_{iy}}{Q_0 + \sum_{i=1}^{N} P_i Q_i} - \frac{2\sum_{i=1}^{N} P_{ix} Q_i [Q_{0y} + \sum_{i=1}^{N} P_i Q_{iy}]}{[Q_0 + \sum_{i=1}^{N} P_i Q_i]^2}, \tag{9}$$

where P_i are arbitrary functions of $\{x,t\}$, Q_0 is an arbitrary function of $\{y-2Bt\}$, and Q_i satisfy (8). Comparing (9) with the universal formula in [1], we find that the expression (9) is an extended universal formula.

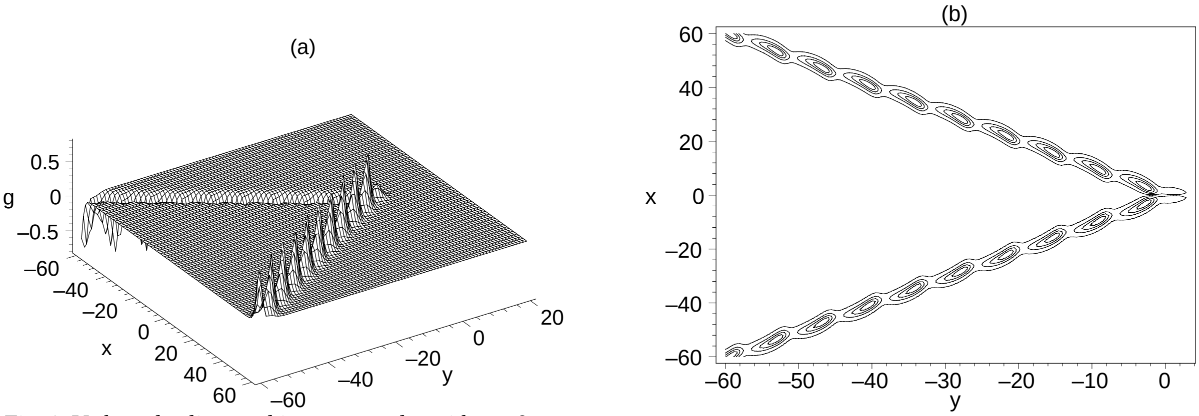

Fig. 1. V-shaped soliton and its contour plot with $t = 0$.

Remark 1. If we consider the simplest case: $N = 1$, $Q_0 = \phi(y - 2Bt)$, $\{P_1, Q_1\} = \{\chi(x,t), 1\}$, then the result in [14] can be recovered.

Remark 2. In a similar way, when we consider the case: $N = 3$, $Q_0 = a_0$, $Q_1 = a_1$, $Q_2 = a_2Q$, $Q_3 = a_3Q$, $a_i =$ const. $(i = 0, \cdots, 3)$, $P_1 = P_3 = P$, $P_2 = 1$ with $P \equiv P(x,t)$ and $Q \equiv Q(y - 2Bt, t)$, (9) is changed into another special exact excitation

$$g = \frac{-2(a_0a_3 - a_1a_2)P_xQ_y}{[a_0 + a_1P + a_2Q + a_3PQ]^2}, \tag{10}$$

which is valid for many other types of known (2+1)-dimensional integrable models like the Davey-Stewartsen (DS) equation, the dispersive long wave (DLW) equation, the Broer-Kaup-Kupershmidt (BKK) system, the Nizhnik-Novikov-Veselov (NVV) equation.

3. Some Novel Localized Structures

We know that based on the special exact solution (10), many localized coherent structures, such as lumps, dromions, peakons, compactons, foldons, ring solitons, were discussed in [1–3, 8, 9, 11]. The properties of peakon-peakon, compacton-compacton, foldon-foldon interactions were studied [1, 2]. In [2, 12, 18], authors investigated the interactions among special semifoldons. However, these localized coherent structures above-mentioned were discussed based on two variable separated functions. To the best of our knowledge, studies on the general solution with more variable separated functions and especially on the V-shaped soliton fusion and Y-shaped soliton fission for the new (2+1)-dimensional GBK system were not reported. In [13], the soliton solution for the Burgers displayed the resonant behaviors. In fact, the resonant soliton solutions are known in both (1+1)-dimensional and (2+1)-dimensional cases. The well-known Y-shaped soliton solutions for the (2+1)-dimensional Kadomtsev-Petviashvili (KP) equation [19] and "spider-web-like" solitons for the coupled KP-DS system [20] are the special (2+1)-dimensional resonant line solitons (multi-solitoffs). In the (1+1)-dimensional case, the resonant soliton solutions display the soliton fission and fusion phenomena [21]. Here we will discuss these special fusion and fission phenomena in the (2+1)-dimensional GBK system.

3.1. V-Shaped Soliton Fusion Phenomenon

Before we discuss the soliton fusion and fission phenomena, we firstly study the novel localized structure, i. e. V-shaped soliton, which is named after its figuration shown in Figure 1. When $n = 2$, $P_1 = Q_1 = 1$, $Q_0 = 0.5\sin(y - t)$, $Q_2 = \exp(y - t)$, $P_2 = \cosh(x)$ in (9), we obtain the V-shaped soliton.

Now we focus our attention on the intriguing fusion phenomenon for the special field g in (2+1)-dimensions, which may exist in certain situations. In Fig. 2, the plots of two V-shaped soliton fusion interactions are revealed for the field g expressed by (9) with the selections

$$P_i = \exp(k_ix - k_i^2t + c_i), \quad Q_0 = 0.5\sin(y - t),$$
$$Q_i = \exp(l_iy - l_i^2t + C_i), \quad \forall i, \tag{11}$$

and

$$N = 5, \quad k_1 = l_1 = c_1 = C_2 = 0,$$

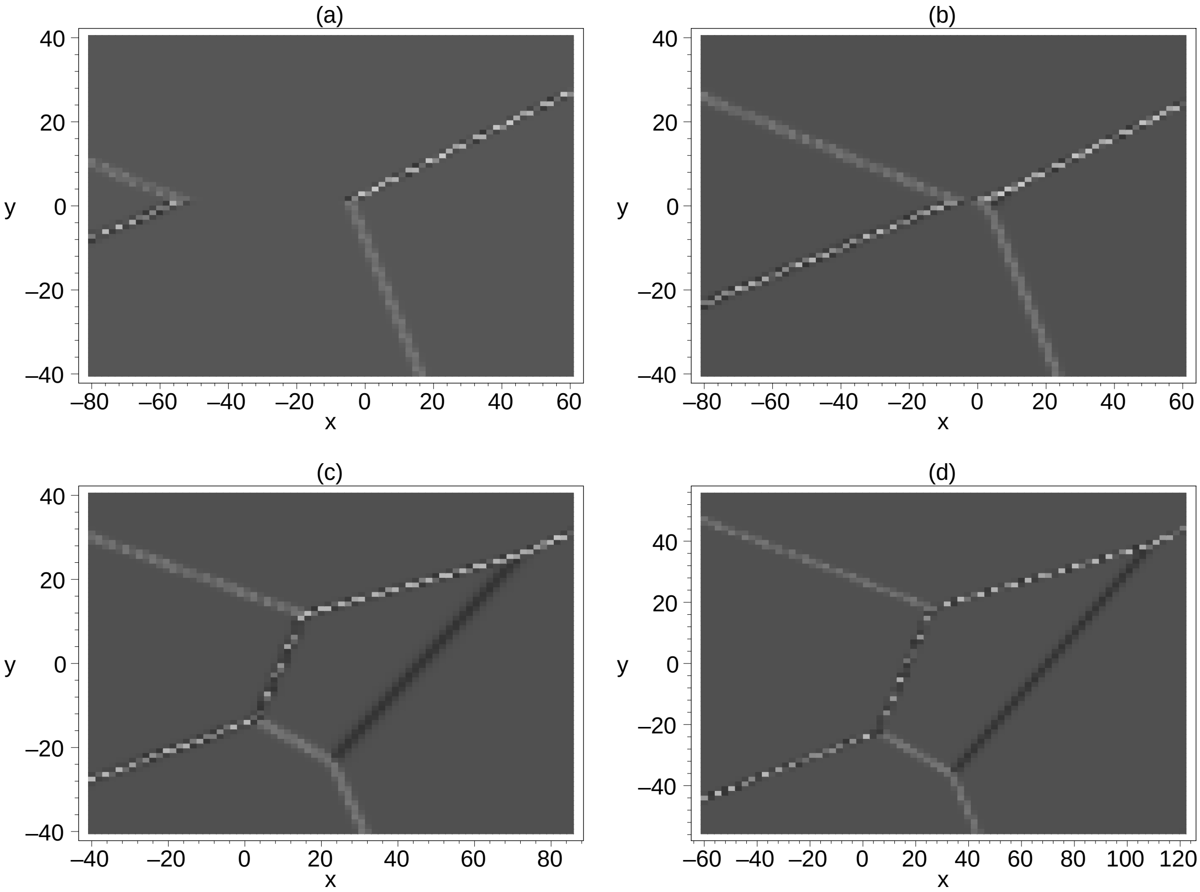

Fig. 2. Plot of the V-shaped soliton fusion interactions for the field g expressed by (9) with (11) and (12) at different times: (a) $t=-5$; (b) $t=-0.4$; (c) $t=5$; and (d) $t=8$.

$$\begin{aligned}
&k_2=k_5=c_2=-l_2=-C_2=1,\\
&k_3=-l_4=2,\\
&k_4=l_2=c_3=C_3=c_5=C_5=-l_5=3,\\
&c_4=C_4=5.
\end{aligned}\tag{12}$$

From Fig. 2, one can see that before interaction, there are two V-shaped solitons. Then, after interaction, the two V-shaped solitons fuse into a single spider-web-like soliton constituted by four Y-shaped solitons.

3.2. Y-Shaped Soliton Fission Phenomenon

In [13], the authors said: "Though we have not yet found the Y soliton fission phenomenon for the (2+1)-dimensional Burgers system, we do believe that it may exist in other (2+1)-dimensional models". It is interesting that for the (2+1)-dimensional GBK system, the resonant Y-shaped solitons may display the soliton fission phenomenon. In Fig. 3, the profiles of Y-shaped soliton fission interactions are displayed for the field g expressed by (9) with the selections (11) and

$$\begin{aligned}
&N=4,\quad k_1=l_1=c_1=C_2=0,\\
&k_2=c_2-C_2=1,\quad k_3=l_4=2,\\
&k_4=-l_2=c_3=C_3=3,\quad c_4=C_4=5.
\end{aligned}\tag{13}$$

From Fig. 3, one can see that the fission phenomenon possesses apparently different evolutional properties compared with fusion phenomena in Fig. 2; the single Y-shaped soliton fissions into three Y-shaped solitons, which constitute a simple spider-web-like soliton.

4. Summary and Discussion

In short, by means of a special Painlevé-Bäcklund transformation and the linear superposition theorem,

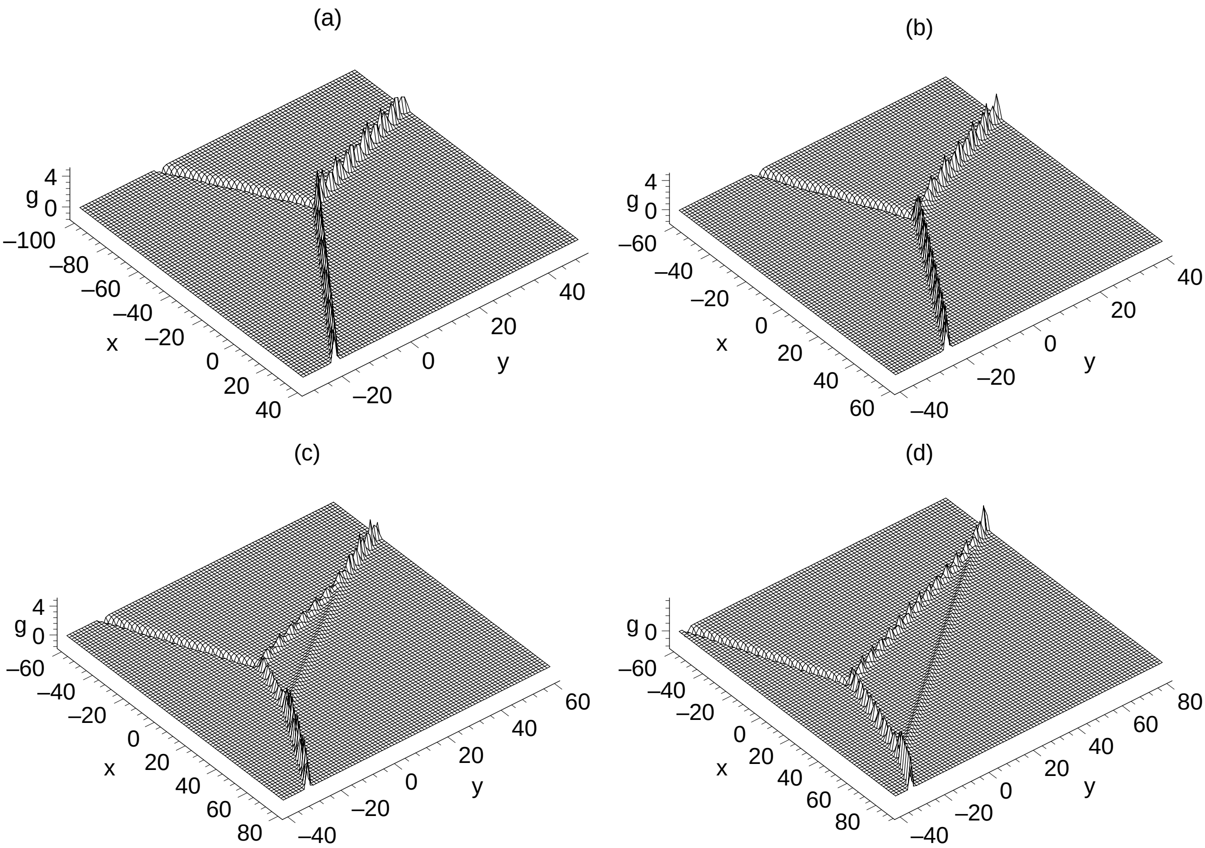

Fig. 3. Profile of the Y-shaped soliton fission interactions for the field g expressed by (9) with (11) and (13) at different times: (a) $t = -8$; (b) $t = -1$; (c) $t = 2$; and (d) $t = 5$.

the general variable separation solution with an arbitrary number of variable separated functions of the GBK system is obtained. Based on the general variable separation solution with some suitable variable separated functions, new types of the V-shaped soliton fusion and Y-shaped soliton fission are firstly discussed.

Though the field g possesses zero boundary conditions for expression (9), different selections of the arbitrary functions P_i, Q_0 and Q_i correspond to different selections of boundary conditions of field with nonzero initial conditions. The changes of these boundary conditions (exotic behavior) propagate along the characteristic and then yield the changes of the localized excitations. That means, in some sense, the fusion and fission phenomena for some physical quantities are remote-controlled by some other quantities which have nonzero boundary conditions. In [22], by using pure numerical calculations, the authors have also pointed out that the nonchaotic dromions of the DS equation can be remote-controlled. From the brief analysis in our present paper, we can see that these intriguing phenomena like the soliton fission and fusion can occur in a higher dimensional soliton system if we choose appropriate initial conditions or boundary conditions, which are similar to some work in (1+1)-dimensions carried out by several authors [22, 23].

Due to the experimental realization of soliton fusion and fission in many physical systems with some special conditions [14 – 17], we think that the discussions here about soliton fusion and fission phenomena in higher dimensional systems are significant and interesting. Clearly, there are still some pending issues to be further studied. How to quantify the notions of soliton fusion and fission phenomena? What are the general equations for the distribution of the energy and momentum for these soliton fusion and fission behaviors? What are the necessary and sufficient conditions for soliton fission and soliton fusion which have been pointed for the (1+1)-dimensional cases in [22]? How can we use the soliton fission and soliton fusion of integrable models to investigate practically observed soliton fission and soliton fusion in the experiments? More of the MLVSA in nonlinear systems, the extended universal formula (9), the soliton fission and fusion phenomena and their applications in reality are worth to be studied further.

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